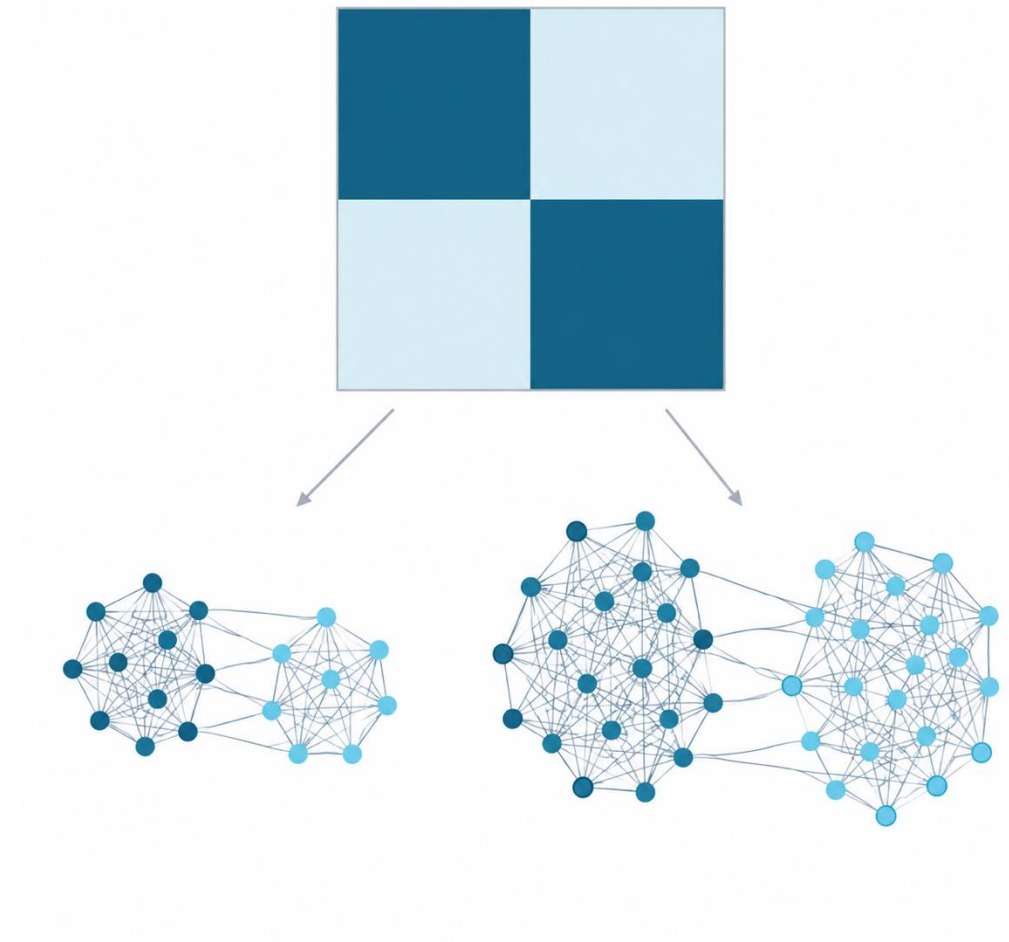


Graphon and graphon signal processing

A continuous model for graph families and signal processing on large graphs

ESE 5291 · Lecture 13



SECTION 1

Graphons

- bounded symmetric measurable function models a family of graphs.
- Sampling node labels and evaluating W generates graphs of any size.
- graphons are limit objects for sequences of graphs

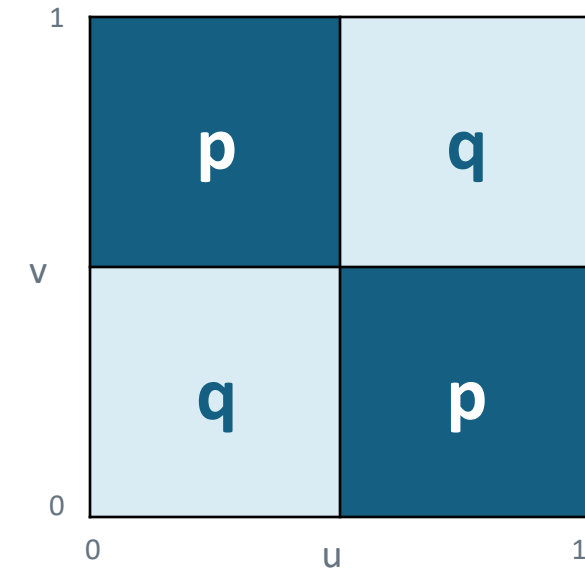
Graphon definition

A graphon is a bounded, symmetric, measurable function.

$$W : [0,1]^2 \rightarrow [0,1]$$

$$W(u, v) = W(v, u)$$

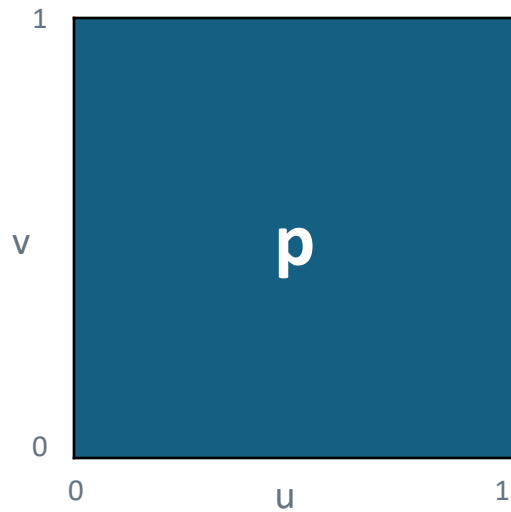
u and v label positions in the population.
 $W(u, v)$ gives their connection strength.



The continuum replaces a finite list of node labels.

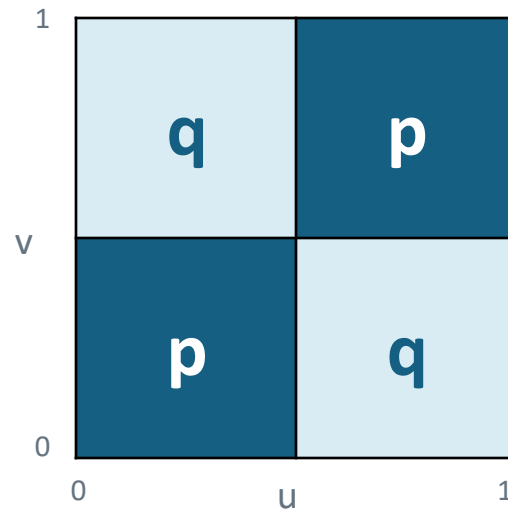
Graphon examples

Uniform (Erdős–Rényi)



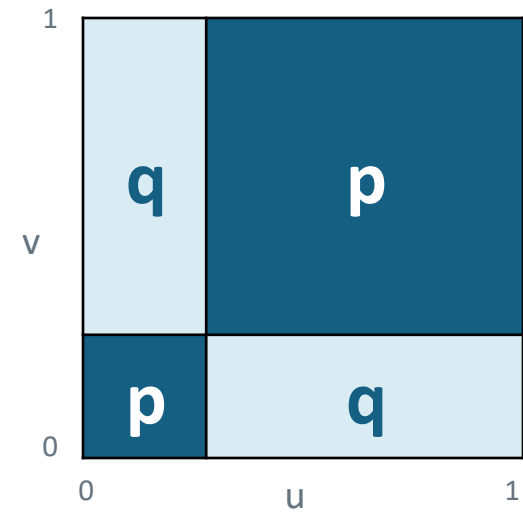
$$W(u, v) = p$$

Balanced SBM



$$p > q$$

Unbalanced SBM



$$p > q$$

Interval width controls group size. Color controls connection strength.

Graphons as generative models

1 SAMPLE NODE LABELS

$$u_1, \dots, u_n \in [0,1]$$

Use a regular grid, or draw independent uniform samples.

2 EVALUATE EACH NODE PAIR

$$S_{ij} = W(u_i, u_j)$$

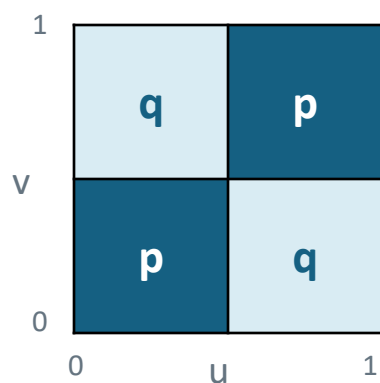
Weighted graph

$$A_{ij} \sim \text{Bernoulli}(W(u_i, u_j))$$

Random unweighted graph

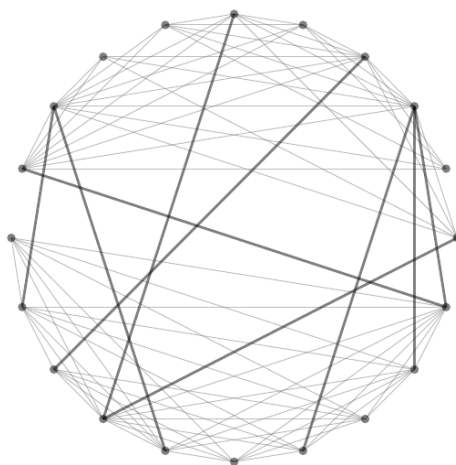
The same rule can generate graphs with different numbers of nodes.

Balanced SBM graphons as generative models

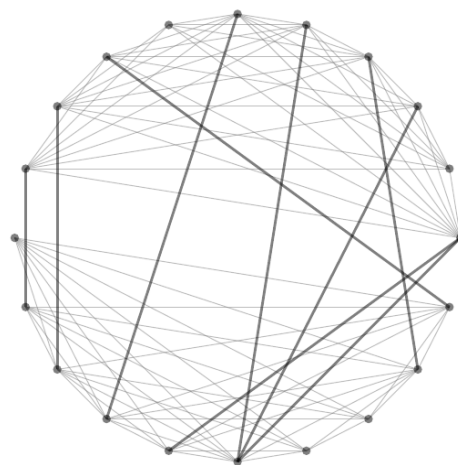


$$P(\text{group 1}) = P(\text{group 2}) = \frac{1}{2}$$

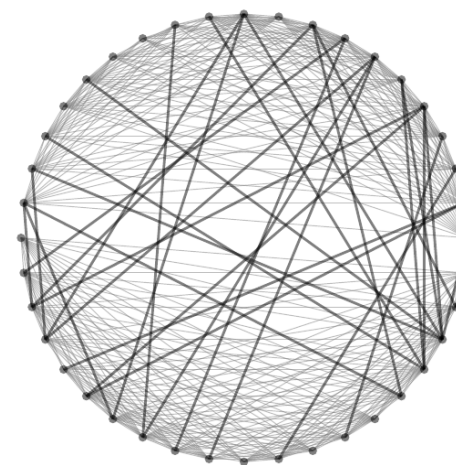
Stronger within-group connections when $p > q$.



n = 20



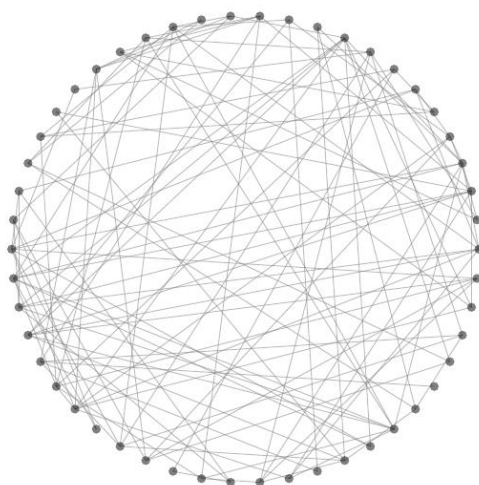
n = 20



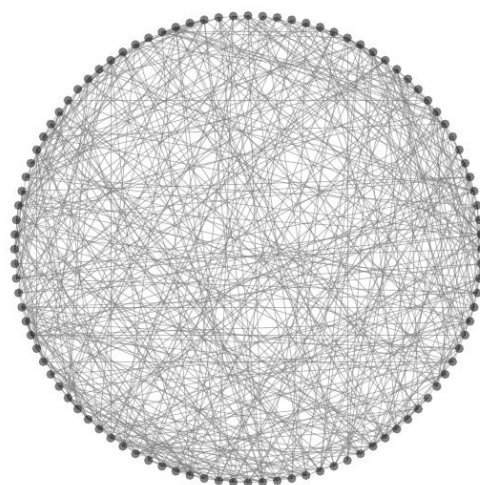
n = 40

Uniform graphons as a limit object

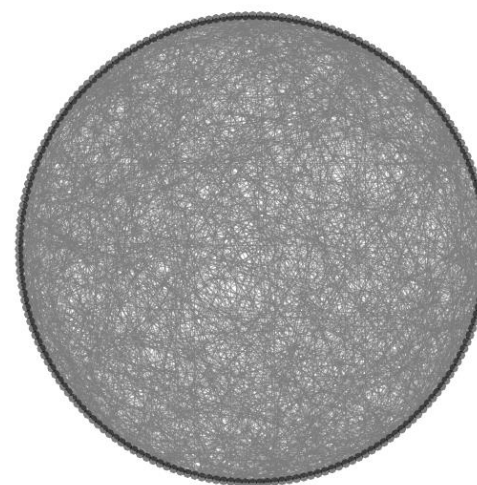
As the number of nodes grows, structural statistics approach a limit.



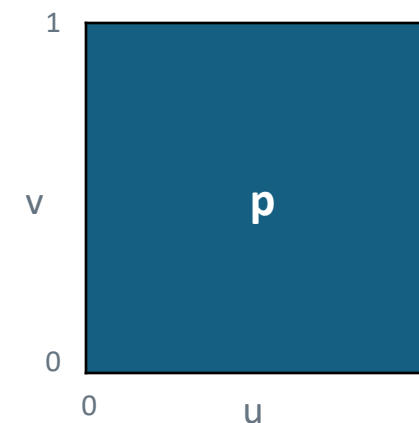
n = 50



n = 100



n = 200



$$W(u, v) = p$$

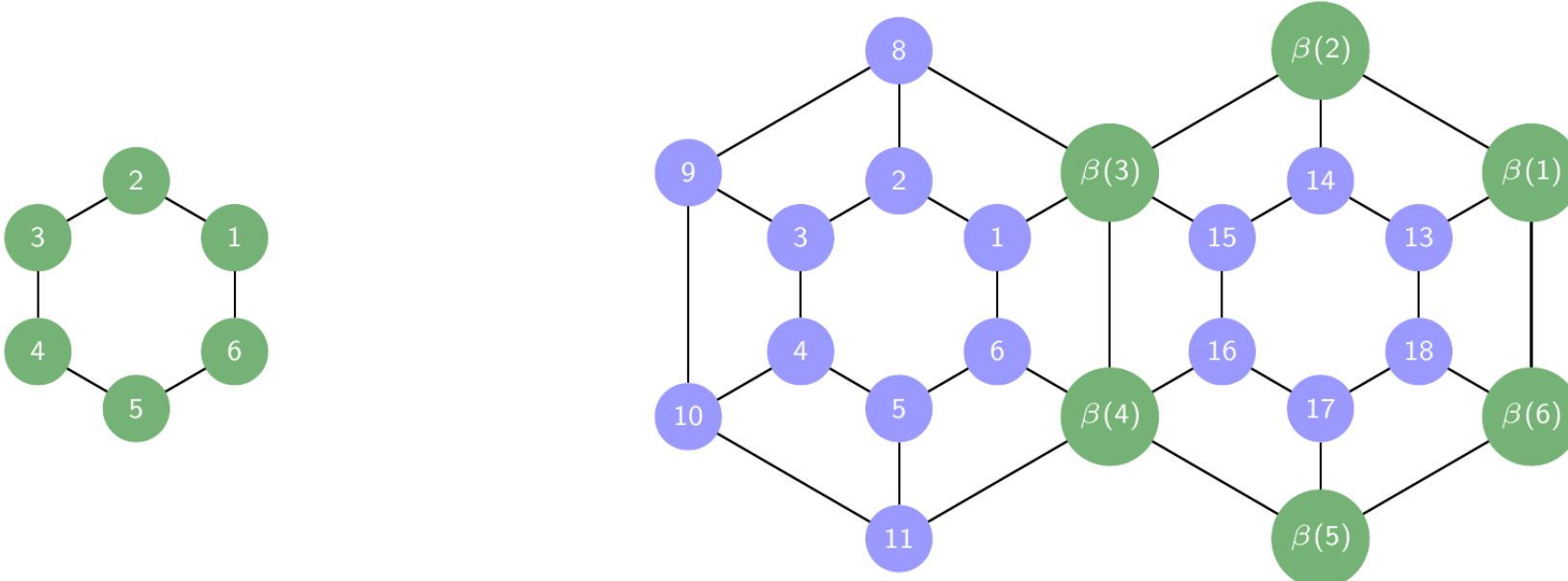
SECTION 2

Convergence of graph sequences

- Motifs ,the homomorphism count and homomorphism density
- A graph sequence converges when homomorphism density converges
- Every graph induces a step-function graphon

Motifs and graph homomorphisms

A motif is a small graph whose adjacency pattern we want to track.



$$\beta : V(F) \rightarrow V(G), \quad (i, j) \in E(F) \Rightarrow (\beta(i), \beta(j)) \in E(G)$$

Preserve edges. Distinct motif vertices do not have to stay distinct.

Homomorphism density

Let the graph have n vertices and the motif have m vertices.

$$t(F, G) = \frac{\text{hom}(F, G)}{n^m} \qquad t(F, G) = \frac{\sum_{\beta} \prod_{(i,j) \in E(F)} [S]_{\beta(i)\beta(j)}}{n^m}$$

SINGLE-EDGE EXAMPLE

$$t(K_2, G) = \frac{2|E(G)|}{n^2}$$

A uniformly chosen ordered pair forms an edge with this probability.

Density makes motif counts comparable across graph sizes.

Homomorphism density for graphons

Average the motif's edge-weight product over all latent labels.

$$t(F, W) = \int_{[0,1]^m} \prod_{(i,j) \in E(F)} W(u_i, u_j) \, du_1 \cdots du_m$$

CONSTANT GRAPHON

$$W(u, v) = p \Rightarrow t(F, W) = p^{|E(F)|}$$

The required edges must appear. Additional edges are allowed.

Convergence in homomorphism density

A sequence of undirected graphs G_n converge to W iff for all motifs F

$$\lim_{n \rightarrow \infty} t(F, G_n) = t(F, W)$$

The convergence holds in **the homomorphism density sense**.

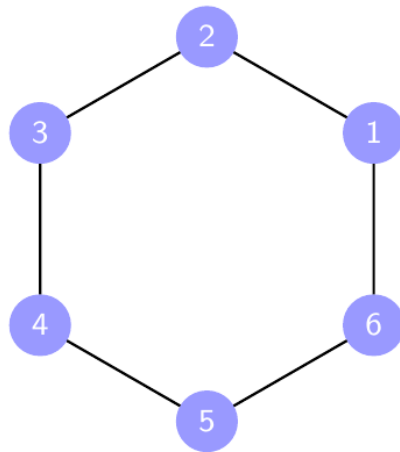
Every graphon is the **limit object** of some convergent graph sequence.

Every convergent graph sequence has **a graphon limit**.

The limit describes structure independently of node labels.

Induced graphons

$$I_i = \left[\frac{i-1}{n}, \frac{i}{n} \right)$$



6-CYCLE

T_{W_G} corresponds to S/n

$$W_G(u, v) = \sum_{i,j=1}^n S_{ij} 1_{I_i}(u) 1_{I_j}(v)$$

1	1	0	0	0	1	0
	0	0	0	1	0	1
	0	0	1	0	1	0
v	0	1	0	1	0	0
	1	0	1	0	0	0
0	0	1	0	0	0	1
	0		u			1

ADJACENCY AS A STEP FUNCTION

SECTION 3

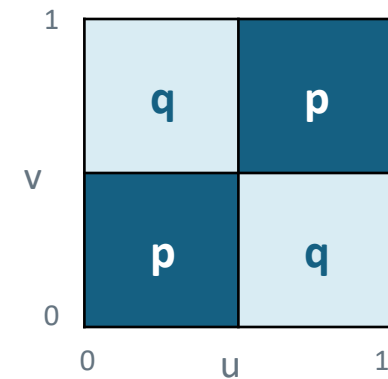
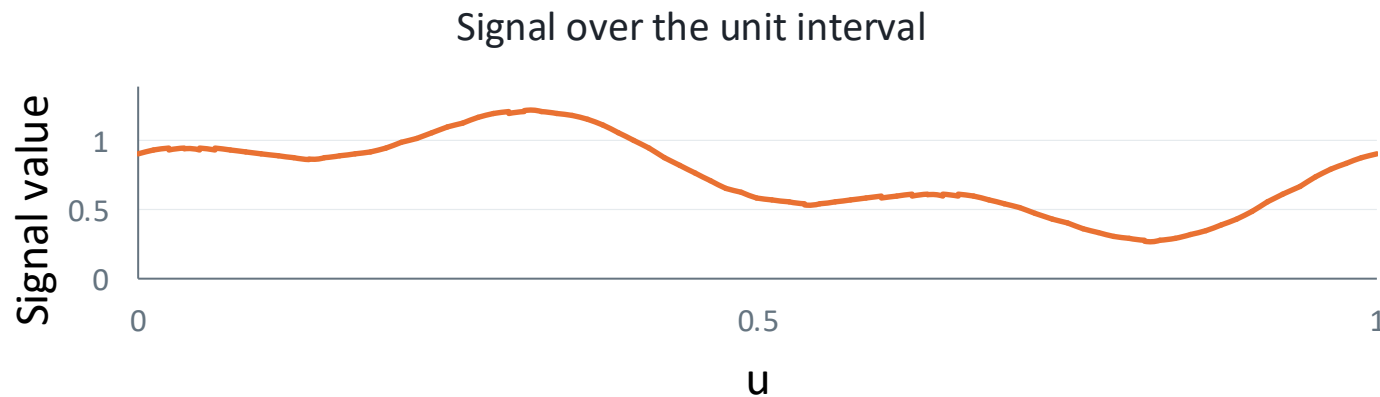
Graphon signals

- A graphon signal is the continuum analogue of a graph signal $x \in \mathbb{R}^n$
- A graph signal induces a step-function signal X_G
- The graphon shift operator diffuses the signal over the graphon

Graphon signals

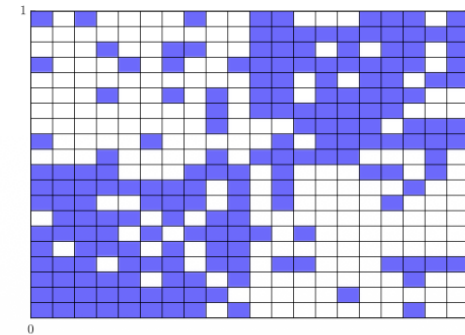
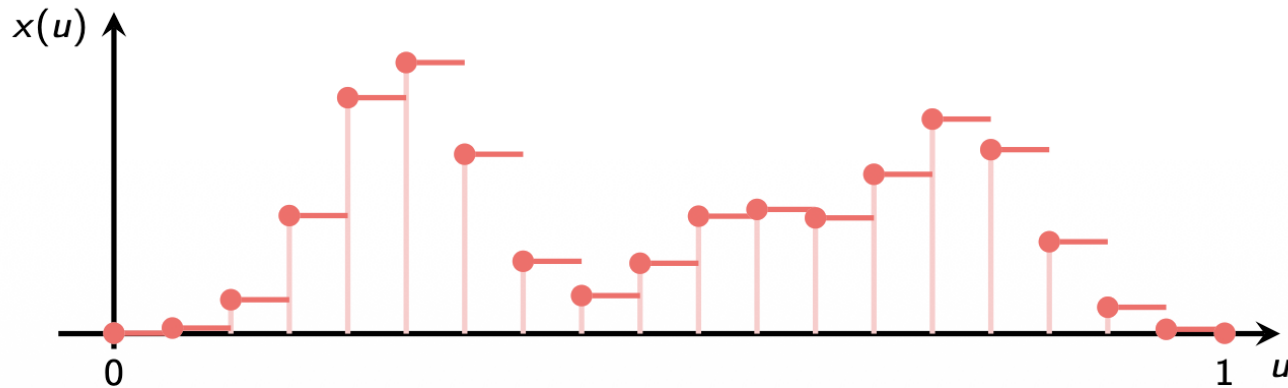
Graphon signal: (W, X) , $X : [0,1] \rightarrow \mathbb{R}$

$$\|X\|_{L^2}^2 = \int_0^1 |X(u)|^2 du < \infty$$



Induced graphon signals

$$I_i = \left[\frac{i-1}{n}, \frac{i}{n} \right) \quad X_G(u) = \sum_{i=1}^n x_i 1_{I_i}(u) \quad \|X_G\|_{L^2}^2 = \frac{1}{n} \sum_{i=1}^n |x_i|^2$$



Graphon shift operator

Integral linear operator
(Hilbert-Schmidt operator)

$$(T_W X)(v) = \int_0^1 W(u, v) X(u) du$$

FINITE GRAPH

$$y_i = \frac{1}{n} \sum_{j=1}^n S_{ij} x_j$$

CONTINUOUS GRAPHON

$$Y(v) = \int_0^1 W(u, v) X(u) du$$

T_W -- Graphon Shift Operator (WSO); Applying T_W to graphon signal diffuse X over graphon W

The graphon shift is weighted aggregation over a continuum.

SECTION 4

Graphon Fourier transform

- The graphon shift T_W is compact and self-adjoint -- eigendecomposition
- Eigenvalues accumulate only at zero; the eigenfunctions form a basis of $L^2[0,1]$
- The WFT projects a signal onto the eigenfunctions and the inverse WFT reconstructs it

Eigenfunctions of the graphon shift

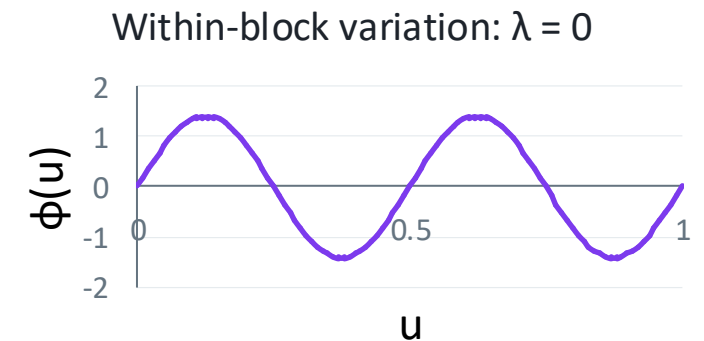
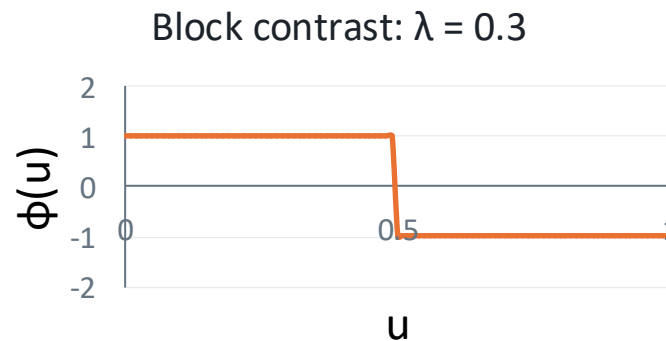
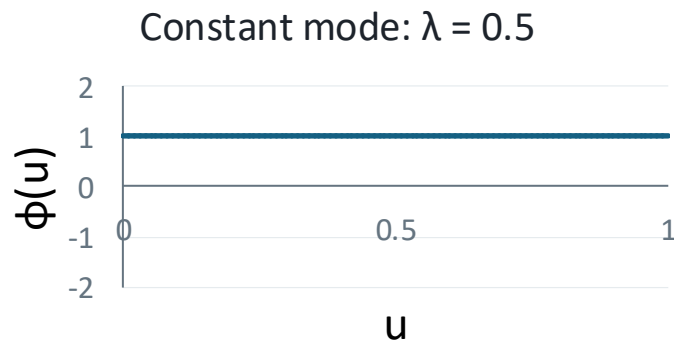
Eigenfunction $\varphi: [0,1] \rightarrow \mathbb{R}$

$$(T_W \varphi_j)(v) = \int_0^1 W(u, v) \varphi_j(u) du = \lambda_j \varphi_j(v)$$

Eigenfunctions are normalized

$$\|\varphi_j\|_{L^2}^2 = \int_0^1 |\varphi_j(u)|^2 du = 1$$

Countable number of eigenvalue–eigenfunction pairs $\{(\lambda_i, \varphi_i)\}_{i=1}^\infty$



Example: balanced SBM with $p = 0.8$ and $q = 0.2$.

Spectral representation of a graphon

Eigenfunctions of T_W as orthonormal basis of $L^2([0,1])$

Decompose graphon W in the basis $\{\varphi_i\}_{i=1}^{\infty}$

$$W(u, v) = \sum_{j \in J} \lambda_j \varphi_j(u) \varphi_j(v)$$

Analogous to eigenvector decomposition

$$S = \mathbf{V} \mathbf{\Lambda} \mathbf{V}^T = \sum_{j=1}^n \lambda_j \mathbf{v}_j \mathbf{v}_j^T$$

Include a basis for the zero-eigenvalue subspace when representing signals.

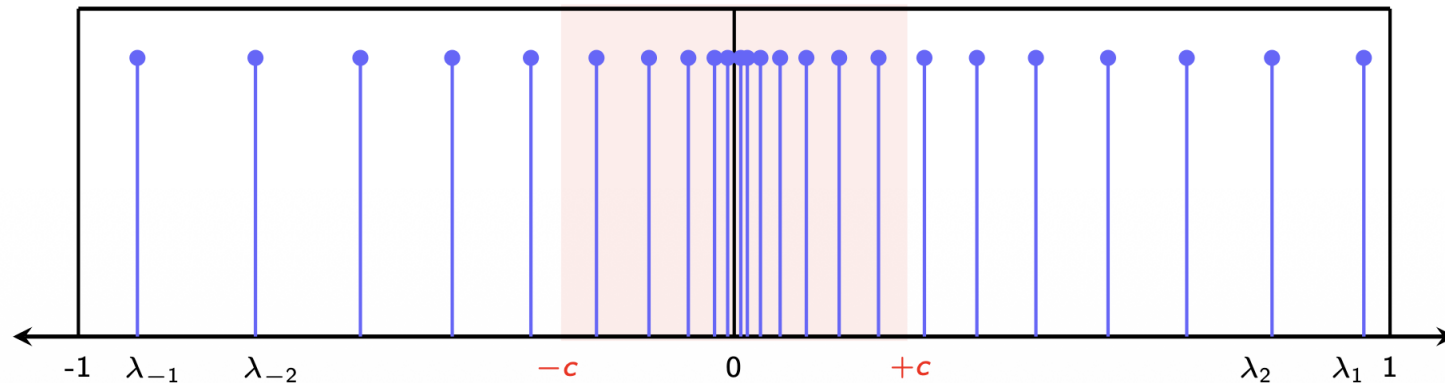
Eigenvalues concentrate around zero

T_W – self adjoint and $0 \leq W(u, v) \leq 1 \Rightarrow$ Eigenvalues are real and lie in $[-1, 1]$

Order as $-1 \leq \lambda_{-1} \leq \lambda_{-2} \leq \dots \leq 0 \leq \dots \leq \lambda_2 \leq \lambda_1 \leq 1$

Eigenvalues accumulate at $\lambda = 0 \Rightarrow \lim_{i \rightarrow \infty} \lambda_i = \lim_{i \rightarrow \infty} \lambda_{-i} = 0$

For any $c > 0$, number of eigenvalues $|\lambda_i| \geq c$ is finite, $n_c < \infty$



Finite-rank graphons are allowed. Infinite nonzero spectra approach zero.

The graphon Fourier transform (WFT)

$$(T_W X)(v) = \int_0^1 W(u, v) X(u) du = \sum_{j=0}^{\infty} \lambda_j \varphi_j(v) \int_0^1 \varphi_j(u) X(u) du$$

Graph Fourier Transform (WFT)

$$\hat{X}_j = \langle X, \varphi_j \rangle = \int_0^1 X(u) \varphi_j(u) du$$

For $j \in \mathbb{Z}/0$

Eigenfunctions can provide a complete representation of X on graphon basis

The inverse graphon Fourier transform

iWFT:
$$X(u) = \sum_{j \in \mathbb{Z}/0} \hat{X}_j \varphi_j(u)$$

Eigenfunctions are orthonormal, the inverse is proper

$$\|X\|_{L^2}^2 = \sum_{j \in J} |\hat{X}_j|^2$$

A complete eigenbasis reconstructs the whole signal, including kernel components.

The graph and graphon Fourier transforms

GFT(G_n, x_n) – eigenspace projection

$$\hat{x}_n(j) = \hat{x}_n(\lambda_{nj}) = \sum_{i=1}^n x_n(i) v_{nj}(i)$$

WFT(W, X) – eigenspace projection

$$\hat{X}_j = \hat{X}(\lambda_j) = \int_0^1 X(u) \varphi_j(u) du$$

Eigenvalues convergence

$$\lim_{n \rightarrow \infty} \frac{\lambda_j(S_n)}{n} = \lim_{n \rightarrow \infty} \lambda_j(T_{W_n})$$

For any convergent graph sequence, the eigenvalues of the graph converge to those of the graphon

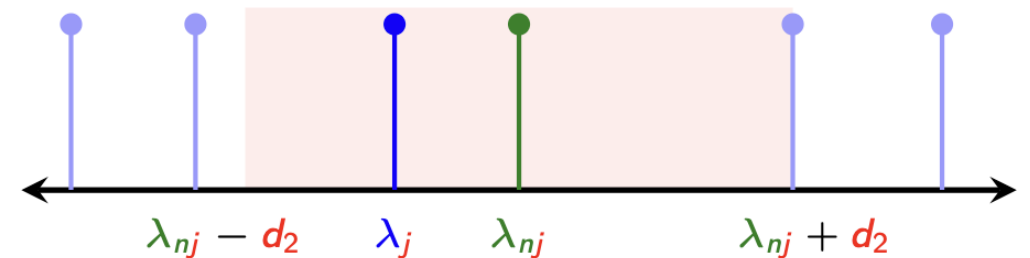
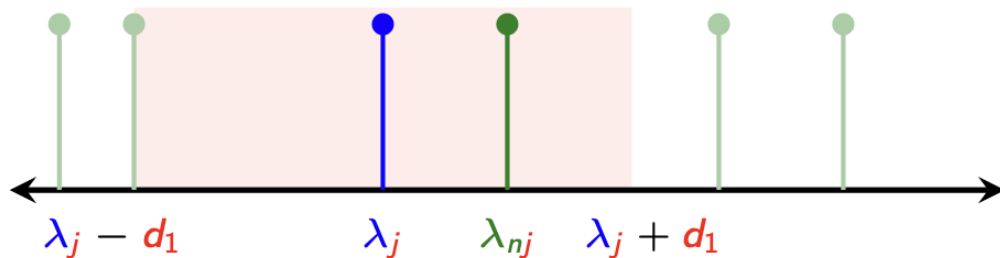
However, the convergence does not hold in general --> convergence of eigenvectors to eigenfunctions

Eigenvector convergence is not uniform

Graph eigenvectors converge to graphon eigenfunctions, how fast depends on the gap to the neighbouring eigenvalues.

$$\text{Davis-Kahan} \quad \|\varphi_j - \varphi_{nj}\| \leq \frac{\pi}{2} \frac{\|W - W_{G_n}\|}{d(\lambda_j, \lambda_{nj})}$$

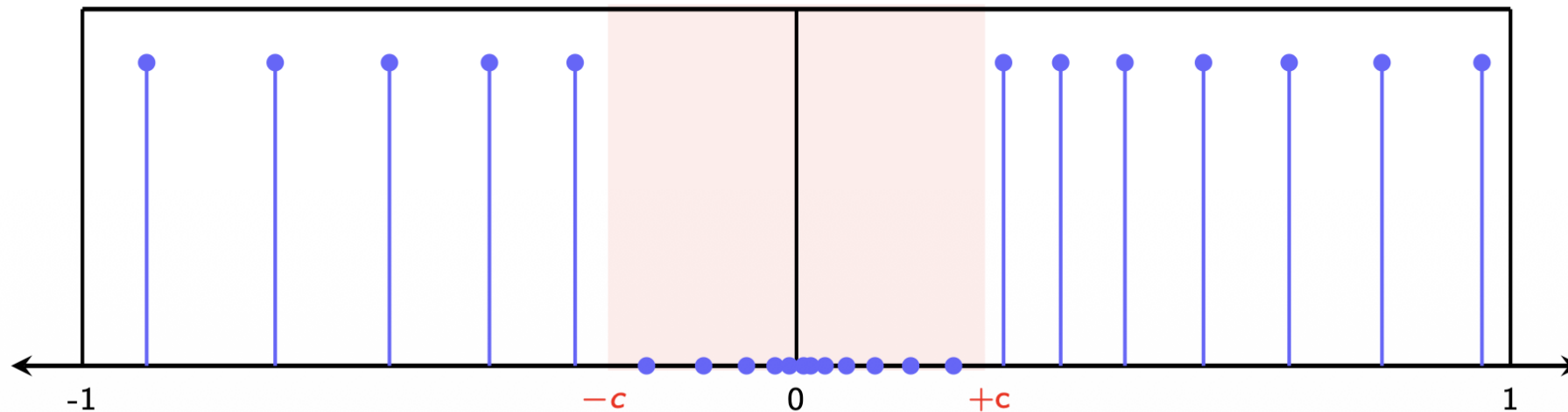
$$d(\lambda_j, \lambda_{nj}) = \min \left(d_1 = \min_{i \neq j} |\lambda_j - \lambda_{ni}|, d_2 = \min_{i \neq j} |\lambda_{nj} - \lambda_i| \right)$$



Eigenvectors converge one at a time, not all at once: the GFT does not converge to the WFT.

Graphon bandlimited signals

c-bandlimited: $\hat{X}(\lambda_j) = 0$ for all $|\lambda_j| < c, c \in (0,1]$



Here “bandlimited” excludes small-magnitude adjacency eigenvalues.

GFT convergence for bandlimited signals

Let (G_n, x_n) be a sequence of graph signals converging to the c-bandlimited graphon signal (W, X) . There exists a sequence of permutations π_n such that

$$GFT(\pi_n(G_n), \pi_n(x_n)) \rightarrow WFT(W, X)$$

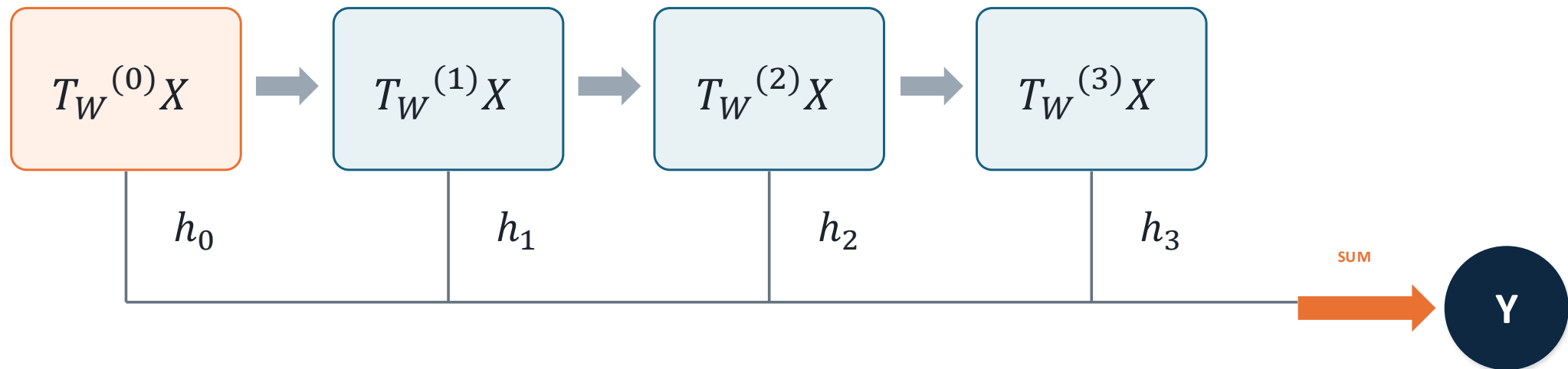
$$\pi_n(iGFT(\hat{x}_n)) \rightarrow iWFT(\hat{X})$$

Bandlimiting and spectral separation give a clean sufficient condition.

Graphon filters and graph filters

The algebra is the same. Replace a matrix shift with an integral shift.

$$\left(T_W^{(k)} X\right)(v) = \int_0^1 W(u, v) \left(T_W^{(k-1)} X\right)(u) du \qquad Y(v) = \sum_{k=1}^K h_k \left(T_W^{(k)} X\right)(v) = (T_H X)(v)$$



The filter uses the same polynomial construction in both settings.

Graphon filters in the Graphon Fourier domain

WFT input signal $\hat{X}_j = \int_0^1 X(u) \varphi_j(u) du$

WFT of output $\hat{Y}_j = \int_0^1 Y(u) \varphi_j(u) du$

Frequency representation of graphon filters

$$\hat{Y}_j = \sum_{k=0}^K h_k \lambda_j^k \hat{X}_j$$

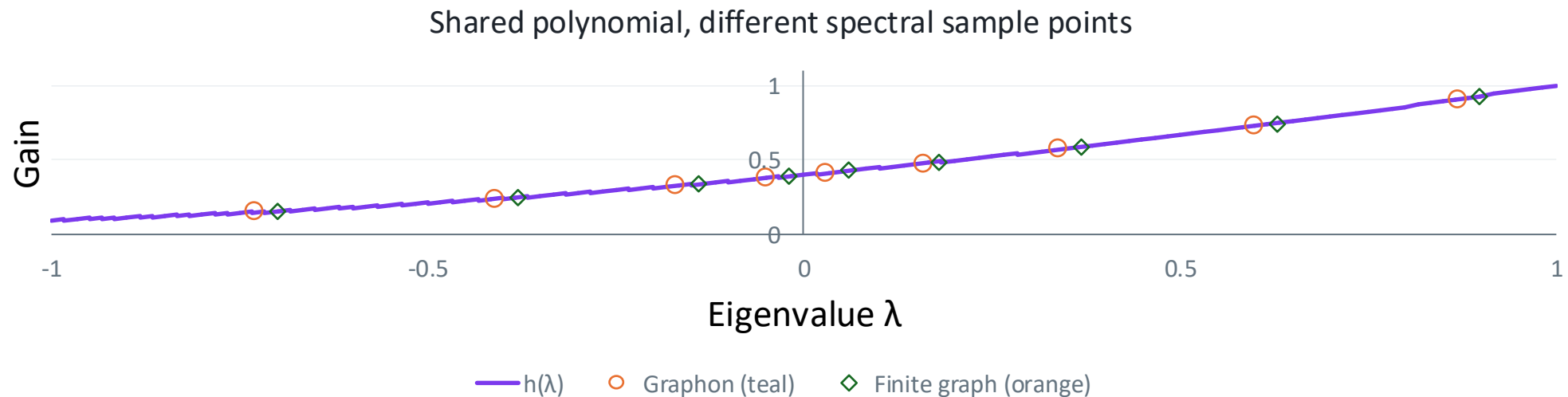
Frequency response of graphon filters

$$h(\lambda) = \sum_{k=0}^{\infty} h_k \lambda^k$$

All analogous to a graph filter with coefficients h_k

Frequency response of graphs and graphons

The same coefficients define the same curve.
Each operator evaluates that curve at its own eigenvalues.



The coefficients define the response. The operator supplies the spectrum.